Statistical Origin of Black Hole Entropy in Slowly Rotating Bumblebee Black Hole Model

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Abstract We show that the entanglement entropy originally defined in the context of quantum mechanics, extended to quantum field theory and now being used in black hole thermodynamics can be extended to spacetime with a lorentz violating field. We study how the presence of such field affects the nature of entanglement entropy in both flat i.e. in the absence of any gravitating body and in the presence of one. It is found that the entanglement entropy converges with the one derived using the arguments from Bekenstein. We discuss two widely used procedure for calculating the entanglement entropy − using replica trick and heat kernel method in such setting. We find that in the presence of non minimal coupling − the heat kernel method needs certain modifications before it can be applied to calculate entropy in such background.

1 Introduction

Entanglement entropy or Geometric entropy has been a topic of interest to both field theorists as well as quantum gravity researchers [1, 16, 7]. It measures the degree of ignorance or lack of information resulting from tracing over the subsystem separated from each other through an entangling surface. It is defined for a system with a membrane, interface or a boundary which divides the whole system into two parts, any causal correlation between them in prevented so that we do not have access to any information about one part of the system. In such condition, the entanglement entropy measures the lack of information resulting from the presence of such boundary. This boundary or membrane which divides the system into two subparts is referred to as entangling surface. When applied to black holes, where the entangling surface is the black hole horizon, the geometrical character of the entanglement entropy becomes crucial.

Especially due to the presence of event horizon which cuts off any causal correlation with the degree of freedom that lies inside the horizon and a vast similarity with

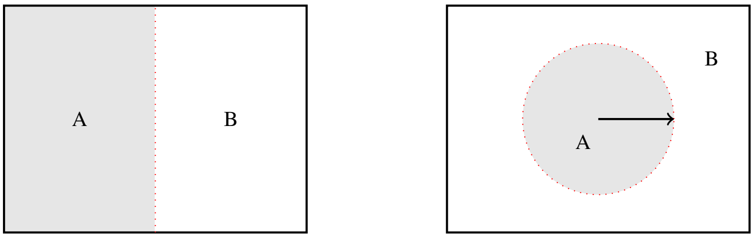

Fig. 1 In the left figure we see the entangling surface denoted as dotted red line divides the system into two complementary parts A and B whereas on the right figure we see the entangling surface denoted as a circle separating the whole system into two complementary parts.

Bekenstein entropy [5]. Due to this property of event horizon as the entangling surface makes it a perfect candidate for studying the nature of entanglement entropy. It was also found that the entanglement entropy depends solely on the geometry of entangling surface and it's embedding in the spacetime. Relevent reviews can be found here [26, 21, 24].

Originally entanglement entropy was studied while preserving the Lorentz symmetry however in recent decade we have begun to question to nature of Lorentz symmetry and its possible violations [27, 17, 20, 28, 6]. This posits a question on the nature of entanglement entropy in the presence of Lorentz violation. In this paper, we wish to study the entanglement entropy of static black hole in the presence of Lorentz violating background known as bumblebee field. In the presence of such field, the Schwarzschild solution takes this form [10]:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + (1+l)\left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

The black hole entropy in bumblebee field is given as [13].

$$dS_{\mathrm{BH}} = \frac{dM}{T_{\mathrm{H}}} = [8\pi M\sqrt{1+l}]dM = d[4\pi\sqrt{1+l}M^2] = \sqrt{1+l}\frac{\mathscr{A}}{4} \tag{1}$$

We expect the entanglement entropy to converge to this form to be a viable candidate for Bekenstein Hawking entropy.

In the recent breakthroughs [2, 3, 23], we have seen that the entanglement entropy becomes very important for describing the unitary evaporation of Black Holes. The so called page curve which describes the entanglement entropy as a function of time has become a central topic for discussion. An essential step in solving Hawking's black hole information paradox is reproducing the Page curve without explicitly assuming unitarity [14]. However it hinges on the assumption that the entanglement entropy is where the Black Hole entropy originates from.

Next, is the issue of renormalization of entanglement entropy [22]. The question weather or not the old treatment on the renormalization should hold depends entirely on the changes induced by Bumblebee field. As we will see in this paper that this is indeed the case.

In our previous work, we explored the entropy for black hole in bumblebee gravity using brick wall method. Where it was found that the black hole entropy becomes independent of the $\sqrt{1+l}$ parameter under certain modifications. In the membrane paradigm of Brickwall method, the entropy of static black hole is given by [19]:

$$\begin{aligned} S &= \tilde{\beta}^2 \frac{\partial F}{\partial \tilde{\beta}} \\ &= \tilde{\beta}^2 \frac{\sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi}{4\pi} \times \frac{\pi}{\lambda \tilde{\beta}^2} \\ &= \frac{\mathscr{A}}{4\lambda} \end{aligned}$$

Clearly, the entropy is not affected by Lorentz violation at all. This happens because the computation of entropy on the membrane is done on a hypershell defined by $dr = 0$ and thus is not affected by g_{rr} term which is the only component of metric tensor to be impacted by the Bumblebee field. It remains to be seen if similar difficulties arise in the study of entanglement entropy as well. However, without studying the entanglement entropy in this setup, it is hard to comment upon the nature of entanglement entropy associated with black holes. These are all the more reason for us to look into the entanglement entropy in Bumblebee gravity.

$$S_{\text{ent}} = -\text{tr}\, \hat{\rho} \ln \hat{\rho}$$

There are two general ways to calculate the entanglement entropy, first is using replica trick which we discuss in section (2) of this paper and another one is using Heat kernel method discussed in section (3). From the literature it is seen that the entanglement entropy is calculated after Euclideanizing the spacetime and then using replica trick to calculate the necessary partition function on $n-$sheeted Riemann manifold for evaluating the entropy [24, 21, 26, 11]. This has an important consequence on the nature of entropy calculated using this method. The process of Identification and Euclideanization imposes the boundary condition on the periodicity of the variable τ. It is this periodicity which governs the form of final result. We wish to show the nature of this identification to explain how the final entanglement entropy changes drastically if one identification is chosen over the other.

2 Entanglement entropy in flat Bumblebee gravity across a hyperplane

In this section we consider the simplest case to begin with by setting the mass parameter of Schwarzschild solution in bumblebee gravity [10] to zero. It is difficult to calculate the density matrix ($\hat{\rho} = e^{-\beta \mathscr{H}}$) directly in QFT because it requires us to know the full spectrum of certain integral operators. This is why the replica trick − a geometric approach − is used to first transform the difficulty of evaluating the density matrix to determining the partition function. General procedure in replica trick is to consider Rényi entropy on $n-$sheeted Riemann manifold obtained by making a cut that begins at the entangling surface and calculate the partition function on the space. We recover the entanglement entropy from Rényi entropy under the limit $n \to 1$. Thus

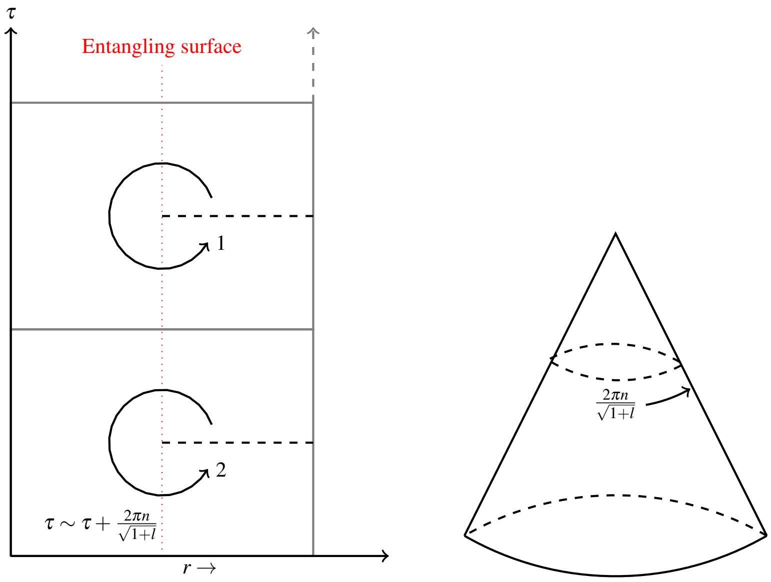

Fig. 2 The vertical red line at $r = 0$ describes the entangling hyperplane which divides the space into two subregions. The cut is made along the $r > 0$ direction at $t = 0$. The identification made on the left leads to a cone that looks like the one described on the right. The trace of ρ^n is calculated over this $n-$ sheeted Riemann manifold.

we begin with the metric[1] and make a cut along $r' > 0$ direction. It changes the metric tensor in following way with a new boundary condition:

$$\begin{aligned} ds^2 &= d\tau'^2 + (1+l)dr'^2 + r'^2 d\Omega^2 \\ &= \underbrace{dr^2 + r^2 d\tau^2}_{C_n} + \underbrace{r'^2 d\Omega^2}_{\mathbb{R}^2} \end{aligned}$$

Here we start with the ansatz that even with the lorentz violating parameter, the identification on τ is invariant [2]. Euclideanization i.e. writing the metric in standard euclidean form $(ds^2 = dr^2 + r^2 d\tau^2)$ makes the angular variable $2\pi/\sqrt{1+l}$ periodic. In canonical ensemble, the entropy on such manifold with tr $\rho^n = \frac{Z_n}{Z_1^n}$ is given by [9, 8]:

$$\begin{aligned} S_{\text{ent}} &= -\text{tr}\hat{\rho}\ln\hat{\rho} = -\partial_n \ln \text{tr}\rho^n\Big|_{n=1} = -\partial_n \ln \frac{Z_n}{Z_1^n}\Big|_{n=1} \\ &= (1-\partial_n)\ln Z_n\Big|_{n=1} \end{aligned} \tag{2}$$

[1] Here neither r is the radial distance nor τ is Euclidean time.
[2] We assume 2π periodicity on r'.

The Z_n is the partition function of scalar field defined on $n-$sheeted Riemann manifold $\mathscr{M}_n$. It is seen that the application of replica trick that the periodicity in τ changes from the $\frac{2\pi}{\sqrt{1+l}}$ to $\frac{2\mathbf{n}\pi}{\sqrt{1+l}}$. The log of partition function can be calculated similar to $Z = (\det[-\nabla^2 + m^2])^{-\frac{1}{2}}$ [12]:

$$\begin{aligned}
\ln Z_n &= -\frac{1}{2}\ln\det(-\nabla^2 + m^2) \\
&= -\frac{1}{2}\mathrm{tr}\ln(-\nabla^2 + m^2) \\
&= \frac{1}{2}\int_{\varepsilon^2}^{\infty}\frac{ds}{s}tr[e^{-s(-\nabla^2+m^2)} - e^{-s}]
\end{aligned} \tag{3}$$

Last step was performed using Schwinger trick [25]

$$\ln M = \int_1^M \frac{dM}{M} = \int_1^M dM \int_0^\infty dse^{-sM} = \lim_{\varepsilon^2\to 0}\int_{\varepsilon^2}^{\infty}\frac{ds}{s}(e^{-s} - e^{-sM})$$

The Euclidean time and the the direction perpendicular to the interface forms the 2-dimensional cone C_n. Since the identification began at the entangling surface all the conic singularities lies on it. In black hole setting this cut on Riemann manifold lies in the radial direction perpendicular to the entangling surface (event horizon) which implies that the trace will be taken along $r \geq 2M$. The manifold $\mathscr{M}_n$ is the direct product of C_n and $\mathbb{R}^2$. This decomposition splits the Laplacian operator into two parts $\nabla^2 = \nabla^2_{C_n} + \nabla^2_{\mathbb{R}^2}$.

In the absence of Lorentz violation i.e. $l = 0$, the rotational symmetry of C_n along the τ direction allows us to fourier decompose the modes $e^{\imath j\tau/n}$ with integer j along the angle τ with period $2n\pi$. The eigenfunctions of $\nabla^2_{C_n}$ with $l = 0$ are given as [15]:

$$\begin{aligned}
\nabla^2_{C_n}\phi_{k,j}(r,\tau) &= -k^2\phi_{k,j}(r,\tau), \qquad (k\in\mathbb{R}^+, j\in\mathbb{Z}) \\
\phi_{k,j}(r,\tau) &= \sqrt{\frac{k}{2\pi n}}e^{\imath j\tau/n}J_{|j/n|}(kr)
\end{aligned}$$

with the normalization condition,

$$\int_{C_n} d^2x\phi_{k,j}(x)\phi^{\star}_{k',j'}(x) = \delta_{jj'}\delta(k-k')$$

On the other hand,

$$\begin{aligned}
\nabla^2_{\mathbb{R}^2}\phi_{k_\perp}(y) &= -k_\perp^2\phi_{k_\perp}(y), \qquad (k_\perp\in\mathbb{R}^+) \\
\phi_{k_\perp}(y) &= \frac{1}{2\pi}e^{\imath k_\perp\cdot y}
\end{aligned}$$

where $1/2\pi$ is the choice of normalization. The only part which changes when we break the symmetry are the eigenvalues of Laplacian on the cone C_n. We evaluate the trace in two parts, one as sum over j and another as integral over k as follows:

$$tr[e^{-s(-\nabla^2+m^2)}] = \int_{C_n} d^2x \sum_{j=-\infty}^{\infty} \int_0^{\infty} dk e^{-s(k^2+m^2)} \phi_{k,j}(x)\phi^\star_{k,j}(x) \cdot \int_{\mathbb{R}^2} d^2y \int d^2k_\perp e^{-sk_\perp^2} \phi_{k_\perp}(y)\phi^\star_{k_\perp}(y)$$

$$= \frac{\mathrm{Vol}(\mathbb{R}^2)}{4\pi s} e^{-sm^2} \int_{C_n} d^2x \sum_{j=-\infty}^{\infty} \int_0^{\infty} dk e^{-s(k^2)} \phi_{k,j}(x)\phi^\star_{k,j}(x) \qquad (4)$$

The $\mathrm{Vol}(\mathbb{R}^2)$ is the surface area of entangling surface. In the presence of black hole where the entangling surface is the event horizon, this volume will be the horizon area $\mathscr{A}$. We can now use (4) for evaluating the $\ln Z_n$ using (3). Note that the equation (4) does not yet have 'n' dependence, that comes from evaluating the integral on the C_n. The divergent $\sum_{j=-\infty}^{\infty}$ will be regularized using $\zeta(s)$ function.

2.1 Eigenvalue of Laplacian on $\mathscr{C}_n$

From the previous section it is clear that the entanglement entropy depends entirely on the eigenfunctions of the Laplacian operator on the cone. Laplacian operator over $\mathscr{C}_n$ with Lorentz violating parameter is given as:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2(1+l)}\frac{\partial^2}{\partial \tau^2}$$
$$\to \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \tau^2}$$

with the Laplacian euclideanized in the last step, we have the eigenvalue problem $\nabla^2\phi = -k^2\phi$. From separation of variables $\phi(r,\tau) = R(r)\Theta(\tau)$:

$$\frac{r^2\partial_{rr}R + r\partial_r R + k^2r^2R}{R} + \frac{\Theta(\tau)''}{\Theta(\tau)} = 0$$

The angular part of the solution has this form:

$$\Theta(\tau)'' = -k_\parallel^2\Theta(\tau)$$
$$\Theta(\tau) = e^{ij\sqrt{1+l}\tau/n}$$

here we have used the boundary condition to specify the topology of space and fixes the form of eigenvalue as $k_\parallel^2 = \frac{j^2}{n^2}(1+l)$. Substituting it back into the PDE, we have:

$$\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{j^2(1+l)}{n^2}\frac{1}{r^2}\right)R(r) = -k^2R(r)$$

This is Bessel's PDE and it has a solution of the form [4]:

$$R(r) = J_{\frac{j\sqrt{1+l}}{n}}(kr)$$

Now that we have the solution, we impose the orthonormality condition:

$$\phi(r,\tau) = N\Theta(\tau)\times R(r) = NJ_{\left|\frac{j\sqrt{1+l}}{n}\right|}(kr)e^{ij\sqrt{1+l}\tau/n} \tag{5}$$

using the result [18]:

$$\int_0^\infty rdrJ_\alpha(ar)J_\alpha(br) = \frac{\delta(a-b)}{a}$$

we have

$$\int_0^\infty rdrJ_\alpha(kr)J_\alpha(kr) = \frac{1}{k} \quad \text{where} \quad \alpha = \frac{j\sqrt{1+l}}{n}$$

This solution can now be used to evaluate the trace needed for partition function defined in (3). Referring back to (4) and using $d^2x = \int_0^{2n\pi/\sqrt{1+l}}\int_0^\infty \sqrt{g}drd\tau$ with $N^2 = \frac{k\sqrt{1+l}}{2n\pi}$.

$$\begin{aligned}
\mathrm{tr}[e^{-s(-\nabla^2+m^2)}] &= \frac{\mathrm{Vol}(\mathbb{R}^2)}{4\pi s}e^{-sm^2}\int_{C_n} d^2x \sum_{j=-\infty}^{\infty}\int_0^\infty dke^{-sk^2}|\phi_{k,j}(x)|^2 \\
&= \frac{\mathrm{Vol}(\mathbb{R}^2)}{4\pi s}e^{-sm^2}\int_{C_n}\frac{2\pi n}{\sqrt{1+l}}rdr\sum_{j=-\infty}^{\infty}\int_0^\infty dke^{-sk^2}|\phi_{k,j}(x)|^2 \\
&= \frac{\mathrm{Vol}(\mathbb{R}^2)}{4\pi s}e^{-sm^2}\frac{\sqrt{1+l}}{12n}
\end{aligned} \tag{6}$$

In the second step we have used the identities:

$$\begin{aligned}
\int_0^\infty dkke^{-sk^2}J_\alpha(kr)^2 &= \frac{e^{-r^2/(2s)}}{2s}I_\alpha\left(\frac{r^2}{2s}\right) \\
\int_0^\infty dr\frac{re^{-r^2/(2s)}}{2s}I_\alpha\left(\frac{r^2}{2s}\right) &= -\frac{\alpha}{2}
\end{aligned} \tag{7}$$

The above two integrals can also be performed with the help of mathematica. We are now left with $\mathrm{tr}(e^{-s}) = \int_{C_n} d^2x\int_{\mathbb{R}^2} d^2ye^{-s}$ which is seen to be proportional to n and thus does not contribute to (2). It can be seen that the equation (7) has UV divergence coming from $\sum_j \alpha$ which can be regularized using

$$2\zeta(-1) = \sum_{j=-\infty}^{\infty}|j| = -\frac{1}{6}$$

Putting it all together and evaluating the integral in 's', we expand the result around $\varepsilon = 0$:

$$\begin{aligned}
S_{\mathrm{ent}} &= (1-\partial_n)\ln Z_n\Big|_{n=1} \\
&= \ln Z_1 - \ln Z_1\partial_n\frac{1}{n}\Big|_{n=1} + \cdots = 2\ln Z_1 + \ldots \\
&= \sqrt{1+l}\frac{\mathscr{A}}{48\pi\varepsilon^2} + \ldots
\end{aligned} \tag{8}$$

Here we see a similar dependence of entropy on $\sqrt{1+l}\mathscr{A}$ as seen in (1) which reassures our confidence in the entanglement entropy. It is an interesting remark to add that this entropy is sensitive to the periodicity imposed upon Euclideanization. If we make a different choice of periodicity in r' such as if $2\pi\sqrt{1+l}$ periodicity was chosen[3], then the order of bessel function would be independent of $\sqrt{1+l}$ and be given by $\alpha = j/n$.

$$ds^2 = \underbrace{d\tau'^2 + (1+l)dr'^2}_{\text{this becomes } C_n} + \underbrace{r'^2 d\Omega^2}_{\mathbb{R}^2}$$

If we evaluate entropy with that boundary condition then the entanglement entropy would not be a viable candidate for explaining the origins of black hole entropy. In the $l \to 0$ limit, we recover original entanglement entropy in 4d [26].

$$S_{\text{ent}} = \frac{\mathscr{A}}{48\pi\varepsilon^2}$$

3 Entropy of the Black Hole

In the previous section we studied the flat spacetime entanglement entropy for scalar field in the presence of Lorentz violating bumblebee field with mass parameter set to zero. In this section we study the entanglement entropy with non zero mass parameter using Heat kernel approach. Consider a more general equation of motion with non minimal coupling:

$$(-\nabla^2 + \xi R + m^2)\phi = 0$$

Heat kernel approach is slightly different from replica trick in terms of how we evaluate the partition function:

$$\ln Z_n = \frac{1}{2}\int_0^\infty \frac{ds}{s} tr e^{-s(\mathscr{D}+m^2)} = \frac{1}{2}\int_0^\infty \frac{ds}{s} \mathrm{tr}\, \mathscr{K}_{M_n}(s) e^{-sm^2}$$

where

$$\mathscr{K}_{M_n}(s) = e^{-s\mathscr{D}} \qquad \text{and} \qquad \mathscr{D} = -\nabla^2 + \xi R$$

Since $\mathscr{K}_{M_n}(s) = e^{-s\mathscr{D}}$ also satisfies the heat equation $(\partial_s + \mathscr{D})\mathscr{K}_{M_n}(s) = 0$ we generally refer to it as heat kernel. The series expansion for trace of heat kernel is given as:

$$\mathrm{tr}\, \mathscr{K}_{M_n} = \frac{1}{(4\pi s)^2}\sum_{i=0}^{\infty} a_i(M_n)s^i$$

Due to the presence of conical singularity on the entangling surface, heat kernel coefficients on $\mathscr{M}_n$ has an expansion of the form:

$$a_i = a_i^{\text{bulk}} + (1-n)a_i^{\mathbb{R}^2} + \dots$$

The space $\mathscr{M}_n$ looks like a direct product of the spaces $\mathscr{C}_n \times \Sigma$ where Σ is the singular entangling surface. This entangling surface in our case is the event horizon $\mathbb{R}^2_{r=r_H}$. From the literature it is seen that the entangling surface area $\mathscr{A}$ term in entropy comes

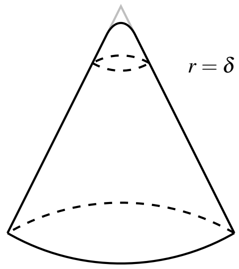

Fig. 3 A cone with a singularity at $r=0$ has been smoothened or regularized by replacing the tip with a disk. We achieve this by adding a smoothing function in the metric.

from surface contribution [26]. Thus we restrict our focus to surface terms which are produced from regularized cone [21]:

$$ds^2_{C_n} = e^{\sigma(r)}[f_\delta(r)dr^2 + r^2(1+l)d\tau^2]$$

smoothing function $f_\delta(r)$ is defined as:

$$f_\delta(r) = \begin{cases} n^2 \text{ if } f_\delta(r \to 0) \\ 1 \ \ \text{ if } f_\delta(r > \delta) \end{cases}$$

The surface term $a_1^{\mathbb{R}^2}$ with non zero l is given as :

$$a_1^{\mathbb{R}^2} = 4\pi(1-n)\sqrt{1+l}\frac{(1-6\xi)}{6}\int_{\mathbb{R}^2} 1$$

The partition function becomes

$$\ln Z_n = \frac{(1-n)\sqrt{1+l}}{2}\int_0^\infty \frac{ds}{4\pi s^2}\frac{\mathrm{Vol}(\mathbb{R}^2)(1-6\xi)}{6}e^{-sm^2} + \dots \tag{9}$$

On comparision with equation (6) we see that both of them has similar structure. It is because heat kernel coefficients have direct dependence on Ricci scalar and higher order curvature tensors, therefore the appearance of $\sqrt{1+l}$ in the $a_1^{\mathbb{R}^2}$ can be easily seen from the integral of Ricci scalar ($\mathscr{R}$) as the coefficient a_1 depends directly on it:

$$\begin{aligned}\int_{C_n}\mathscr{R} &= 2\pi n\sqrt{1+l}\int_0^\infty dr[-2\partial_r f_\delta^{-1/2}(r) - r f_\delta^{-1/2}(r)\partial_r^2\sigma] \\ &= 4\pi n\sqrt{1+l}(1-n) - \int_{C_n} f_\delta^{-1/2}(r)\partial_r^2\sigma\end{aligned}$$

An interesting remark to be made here is that Ricci scalar on cone $\mathscr{C}_n$ stays same even in the presence of Lorentz violating Bumblebee field, however, the induced metric h_{ij} on the cone changes drastically. All the conic singularities lie on the entangling

[3] Instead of 2π

surface ($\mathbb{R}^2$), which is regularized with a smoothing function to calculate the surface term. In the limit $l \to 0$ we recover the original result [21].

$$\int_{C_n} \mathscr{R} = 4\pi n(1-n) - \int_{C_n} f_\delta^{-1/2}(r)\partial_r^2 \sigma$$

The partition function in equation (9) can be used to evaluate the entropy for black hole.

$$S_{\text{ent}} = \sqrt{1+l}\,\frac{(1-6\xi)\mathscr{A}}{48\pi\varepsilon^2} + \dots$$

In minimal coupling setup $\xi = 0$, we recover (8).

$$S_{\text{ent}} = \sqrt{1+l}\,\frac{\mathscr{A}}{48\pi\varepsilon^2} + \dots$$

4 Discussion and Conclusion

In this paper we studied the concept of entanglement entropy on hyperplane as well as in semi classical gravity with spontaneously broken Lorentz symmetry. We recovered a similar expression to brickwall calculations and one which can be found from Bekenstein [5, 13] and found that short distance cutoff ε was unaffected by symmetry breaking. This leads to an interesting conclusion that the renormalization procedure outlined by [22] works exactly the same way with a slight modification in the order of Bessel function. It was seen that the replica trick is sensitive to the Euclidean periodicity in τ and improper choice of periodicity can lead to wrong result. We also studied the entropy using Heat Kernel procedure and found certain modifications to Heat kernel coefficients from Lorentz violation. We regained some more confidence in entanglement entropy after Heat kernel method gave us the same result. Some caveats to the problem stays, the form of metric tensor for a rotating black hole in bumblebee gravity is rather non trivial. In such condition the Euclideanization of metric becomes more complicated and thus the heat kernel approach or replica trick could result unfaithful. It should be noted that in bumblebee field, Ricci scalar doesn't vanish unlike Schwarzschild solution. This should produce some higher order corrections to the entropy.

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